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RATE OF DECAY OF SOME PETROWSKY-LIKE DISSIPATIVE SYSTEMS

KAÏS AMMARI, MOUEZ DIMASSI, AND MAHER ZERZERI

ABSTRACT. In this paper, we show that the fastest decay rate for some Petrowsky-like dissipative systems is given by the supremum of the real part of the spectrum of the infinitesimal generator of the underlying semigroup, if the corresponding operator satisfied some spectral gap condition. We give also some applications to illustrate our setting.

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1. INTRODUCTION AND MAIN RESULT

The determination of optimal decay rate is difficult and has not a complete answer in the general case. In the 1-d case, it was performed mostly, see [1, 2, 5, 6, 7, 8, 9], and to references therein. For higher dimension, G. Lebeau gives in [13] the explicit (and optimal) value of the best decay rate in terms of the spectral abscissa of the generator of the semigroup and the mean value of a damping coefficient along the rays of geometrical optics.

In this paper, we describe, in abstract setting, the optimal decay rate for some Petrowsky-like dissipative systems in terms of spectral quantity of the corresponding infinitesimal generator of the underlying dynamic. The main idea is to identify the optimal energy decay rate with the supremum of the real part of the associated dissipative operator. To do this, it is enough to show that the set of the corresponding generalized eigenvectors forms a Riesz basis of the energy space. The approach used here is based on some resolvent estimates which is obtained by a perturbative method.

In case of the damped wave operator, Cox and Zuazua ([7]) adopt the shooting method based on an ansatz of Horn. This approach consists in constructing an explicit approximation of the characteristic equation of the underlying system. Under the assumption that the damping is of bounded variation, they obtained high frequency asymptotic expansions of the spectrum. The shooting method can be used only for one-dimensional boundary value problems.

In the both cases, we require precise knowledge of the spectrum of the corresponding non self-adjoint operators, more precisely, the behavior of the high frequency set. The advantage of our approach is that it works in any dimension and in a very general setting (see [15] and also [12]).

Let us introduce the abstract setting. Let H be a Hilbert space equipped with the norm $\|\cdot\|_H$. Let A be an unbounded operator on H , self-adjoint, positive and with compact inverse. We denote its domain by $\mathcal{D}(A)$.

Let B be a bounded operator from U to H , where $(U, \|\cdot\|_U)$ is another Hilbert space which will be identified with its dual.

We consider the following system:

$$\begin{cases} \ddot{x}(t) + Ax(t) + BB^*\dot{x}(t) = 0, \\ (x(0), \dot{x}(0)) = (x_0, x_1) \in H_{\frac{1}{2}} \times H, \end{cases} \quad (1.1)$$

where $t \in [0, \infty)$ is the time and $H_{\frac{1}{2}} = \mathcal{D}(A^{\frac{1}{2}})$ the scaled Hilbert space with the norm $\|z\|_{\frac{1}{2}} = \|A^{\frac{1}{2}}z\|_H$, $\forall z \in H_{\frac{1}{2}}$.

From now on, we set $\mathcal{H} := H_{\frac{1}{2}} \times H$. We endowe this space with the inner product:

$$\left\langle [f, g], [u, v] \right\rangle_{\mathcal{H}} := \langle A^{\frac{1}{2}}f, A^{\frac{1}{2}}u \rangle_H + \langle g, v \rangle_H, \quad \text{for all } [f, g], [u, v] \text{ in } \mathcal{H}.$$

We can rewrite the system (1.1) as a first order differential equation, by putting $Y(t) = {}^T(x(t), \dot{x}(t))$:

$$\begin{cases} \dot{Y}(t) + \mathcal{A}_B Y(t) = 0, \\ Y(0) = {}^T(x_0, x_1) \in \mathcal{H}, \end{cases} \quad (1.2)$$

where $\mathcal{A}_B := \mathcal{A}_0 - \mathcal{B} : \mathcal{D}(\mathcal{A}_B) = \mathcal{D}(\mathcal{A}_0) \subset \mathcal{H} \rightarrow \mathcal{H}$, with

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} : \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(A) \times H_{\frac{1}{2}} \subset \mathcal{H} \rightarrow \mathcal{H},$$

and $\mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & BB^* \end{pmatrix} \in \mathcal{L}(\mathcal{H})$.

The operator \mathcal{A}_0 is skew-adjoint on \mathcal{H} hence it generates a strongly continuous group of unitary operators on \mathcal{H} , denoted by $(\mathbf{S}_0(t))_{t \in \mathbb{R}}$. Since \mathcal{A}_B is dissipative and onto, it generates a contraction semi-group on \mathcal{H} , denoted by $(\mathbf{S}_B(t))_{t \in \mathbb{R}^+}$. The system (1.1) is well-posed. More precisely, the following classical result holds.

Proposition 1.1. *Suppose that $(x_0, x_1) \in \mathcal{H}$. Then the problem (1.1) admits a unique solution $t \mapsto x(t)$ in the space $C([0, +\infty); H_{\frac{1}{2}}) \cap C^1([0, +\infty); H)$. Moreover the solution $t \mapsto x(t)$ satisfies the following energy identity:*

$$E(x(0)) - E(x(t)) = \int_0^t \|B^* \dot{x}(s)\|_U^2 ds, \quad \text{for all } t \geq 0, \quad (1.3)$$

where $E(x(t)) = \frac{1}{2} \left\| (x(t), \dot{x}(t)) \right\|_{\mathcal{H}}^2$.

From (1.3) it follows that the mapping $t \mapsto \left\| (x(t), \dot{x}(t)) \right\|_{\mathcal{H}}^2$ is non-increasing. In many applications it is important to know if this mapping decays exponentially when $t \rightarrow +\infty$, i.e., if the system (1.1) is exponentially stable. One of the methods currently used for proving such exponential stability results is based on an observability inequality for the conservative system associated to the initial value problem

$$\ddot{\phi}(t) + A\phi(t) = 0, \quad (\phi(0), \dot{\phi}(0)) = (x_0, x_1) \in \mathcal{H}. \quad (1.4)$$

It is well-known that (1.4) is well-posed in \mathcal{H} . The result below, proved in [11] (see also [3]), shows that the exponential stability of (1.1) is equivalent to an observability inequality for (1.4).

Proposition 1.2. *The system described by (1.1) is exponentially stable in \mathcal{H} if and only if there exist $T > 0$, and $C_T > 0$ such that*

$$C_T \int_0^T \left\| \begin{pmatrix} 0 & B^* \end{pmatrix} \mathbf{S}_0(t) Y_0 \right\|_U^2 dt \geq \|Y_0\|_{\mathcal{H}}^2, \quad \text{for all } Y_0 \in \mathcal{H}. \quad (1.5)$$

The spectrum of A is given by $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots \leq \mu_n \leq \dots \rightarrow +\infty$ and the family $(v_n)_{n \geq 1}$ of corresponding normalized eigenvectors of A is an orthonormal basis of H . Now, we can describe the spectrum of the skew-adjoint operator \mathcal{A}_0 by the following:

Lemma 1.3. *The eigenvalues of \mathcal{A}_0 and the corresponding eigenvectors are given by:*

$$\mathcal{A}_0 V_{\pm k} = (\pm i \sqrt{\mu_k}) V_{\pm k}, \text{ where } V_{\pm k} = \frac{v_k}{\sqrt{2}} \left[\frac{1}{\sqrt{\mu_k}}, \pm i \right], \text{ for all } k \in \mathbb{N}^*. \quad (1.6)$$

Moreover, the family $(V_{\pm k})_{k \in \mathbb{N}^*}$ is an orthonormal basis of the energy space \mathcal{H} .

1.1. Main result. As mentionned in the above, we give the value of the fastest decay rate of solution of the equation (1.1), in terms of the spectral abscissa of the generator \mathcal{A}_B .

Let $\mu(B)$ be the *spectral abscissa* of \mathcal{A}_B given by:

$$\mu(\mathcal{A}_B) = \sup \{ \operatorname{Re}(\lambda); \lambda \in \sigma(\mathcal{A}_B) \}. \quad (1.7)$$

Here $\sigma(\mathcal{A}_B)$ denotes the spectrum of \mathcal{A}_B . In order to state the result on the optimal decay rate, we define the decay rate, depending on B , as

$$\omega(B) = \inf \{ \omega; \text{ there exists } C = C(\omega) > 0 \text{ such that}$$

$$E(x(t)) \leq C(\omega) e^{2\omega t} E(x(0)) \text{ for every solution of (1.1) with initial data in } \mathcal{H} \}. \quad (1.8)$$

According to (1.3), $\omega(B) \leq 0$ (see [11] and also [3]). It follows easily that,

$$\mu(\mathcal{A}_B) \leq \omega(B). \quad (1.9)$$

The following assumption concern the high frequencies of \mathcal{A}_0 . More precisely, it deals with the behavior of the gap between two consecutive high frequencies of \mathcal{A}_0 . For $k \in \mathbb{N}^*$, we define $\delta_{\pm k} := |\pm i(\sqrt{\mu_{k+1}} - \sqrt{\mu_k})| = \sqrt{\mu_{k+1}} - \sqrt{\mu_k}$. We assume that

$$\textbf{(A1)} \quad \lim_{k \rightarrow +\infty} \delta_k = +\infty,$$

and

$$\textbf{(A2)} \quad \left(\frac{\delta_{k+1}}{\delta_k^2} \right)_{k \geq 1} \in l^2(\mathbb{N}^*), \text{ where } l^2(\mathbb{N}^*) \text{ is the space of square integrable sequences.}$$

Remarks

(i) The assumption **(A1)** implies that the high frequencies of \mathcal{A}_0 are simple.

(ii) Assumption **(A2)** implies

$$\lim_{k \rightarrow +\infty} \left(\frac{\delta_{k+1}}{\delta_k^2} \right) = 0. \quad (1.10)$$

(iii) Note that, in general, assumption **(A2)** does not imply hypothesis **(A1)**.

Now, our main result on the optimal decay rate is:

Theorem 1.4. *Assume **(A1)** and **(A2)**. Then,*

$$\omega(B) = \mu(\mathcal{A}_B). \quad (1.11)$$

In other words if all finite energy solutions of (1.1) are exponentially stable then the fastest decay rate of the solution of (1.1) satisfies (1.11).

Outline of the proof: In the following, we give an idea of the proof of the main result.

First, for $B = 0$, the operator \mathcal{A}_0 is skew-adjoint with compact resolvent in \mathcal{H} . From general operator theory, all its eigenvalues lie on the imaginary axis and the geometric and algebraic multiplicity of each eigenvalue are the same. Moreover, there is a sequence of eigenvectors of \mathcal{A}_0 which forms a Riesz (orthonormal, actually) basis for \mathcal{H} .

In our setting, i.e., $B \in \mathcal{L}(U, H)$, we give in Proposition 2.1 rough preliminary bounds on the spectrum of \mathcal{A}_B . Moreover, since \mathcal{A}_B is a bounded perturbation of skew-adjoint operator with compact resolvent it follows from [10, Chapter 5, Theorem 10.1] that the generalized eigenvectors of \mathcal{A}_B are complete in \mathcal{H} . For instance, these results are not enough to prove Theorem 1.4. We need to study the high frequency of \mathcal{A}_B , and in particular their algebraic multiplicities. Using the fact that the distance between two consecutive eigenvalues tends to infinity at infinity, as well as the fact that the dissipation is bounded, we construct in Subsection 2.1 a closed curves $(\Gamma^{(k)})_{|k| > N_0}$ (for some integer N_0 sufficiently large) in the complex plane such that:

- (i) For all $n \in \mathbb{N}^*$, $\Gamma^{(\pm n)}$ is centered in $(\pm i\sqrt{\mu_n})$.
- (ii) Inside each $\Gamma^{(n)}$ there exists exactly one simple eigenvalue of \mathcal{A}_B .
- (iii) The operator \mathcal{A}_B has exactly $2N_0$ eigenvalues including multiplicity in $\mathbb{C} \setminus (\bigcup_{|k| > N_0} \Gamma^{(k)})$.
- (iv) $\sum_{|k| > N_0} \|P_{\Gamma^{(k)}}^B - P_{\Gamma^{(k)}}^0\|_{\mathcal{L}(\mathcal{H})}^2 < \infty$, where $P_{\Gamma^{(k)}}^B$ (resp. $P_{\Gamma^{(k)}}^0$) denotes the Riesz projection associated to \mathcal{A}_B (resp. \mathcal{A}_0) corresponding to $\Gamma^{(k)}$.

The proof of the above statements are based on some resolvent estimates of the operators \mathcal{A}_B and \mathcal{A}_0 . Since the generalized eigenvectors of \mathcal{A}_B are complete and the systems of (generalized)-eigenvectors of \mathcal{A}_B and \mathcal{A}_0 are quadratically close in \mathcal{H} (see (iv) above), it follows from [14, Appendix D, Theorem 3] that the system of generalized eigenvectors of \mathcal{A}_B constitutes a Riesz basis in \mathcal{H} . Now, by a standard argument, we identify the optimal energy decay rate with the supremum of the real part of \mathcal{A}_B , which complete the proof of Theorem 1.4.

2. PROOF OF THE MAIN RESULT

As indicated in the introduction, we will establish Theorem 1.4 by proving that the system of generalized eigenvectors of the operator \mathcal{A}_B constitutes a Riesz basis in the energy space \mathcal{H} , and that all eigenvalues of \mathcal{A}_B with sufficiently large modulus are algebraically simple.

2.1. Description of the spectrum of \mathcal{A}_B . The operator \mathcal{A}_B is a bounded perturbation of a skew-adjoint operator \mathcal{A}_0 then, according to [10, Chapter 5, Theorem 10.1], we have the following spectral result:

Proposition 2.1. *The following properties hold:*

- i) *The resolvent of \mathcal{A}_B is compact. In particular the spectrum of \mathcal{A}_B is discrete, i.e., \mathcal{A}_B has a discrete eigenvalues of finite algebraic multiplicity.*
- ii) *The spectrum of \mathcal{A}_B is symmetric about the real axis and is contained in $\mathcal{C} \cup \mathcal{I}$, where*

$$\mathcal{C} = \left\{ \lambda \in \mathbb{C}; |\lambda| \geq \sqrt{\mu_1}, -\beta \leq \operatorname{Re}(\lambda) \leq 0 \right\} \quad (2.1)$$

$$\mathcal{I} = \left[-\beta - (\beta^2 - \mu_1)_+^{\frac{1}{2}}, (\beta^2 - \mu_1)_+^{\frac{1}{2}} \right]. \quad (2.2)$$

Here $\beta := \frac{1}{2} \|B^*\|_{\mathcal{L}(H,U)}^2 < +\infty$, $\mu_1 > 0$, is the first eigenvalue of A and $(\gamma)_+ = \max(\gamma, 0)$.

- iii) *The root vectors of \mathcal{A}_B are complete in \mathcal{H} .*

Proof. We give only the proof of the second point of the proposition. Let $\lambda_k := \lambda_k(\mathcal{A}_B)$ be an eigenvalue of \mathcal{A}_B . We denote by $W(\cdot; \lambda_k)$ the corresponding eigenvector. Then $W(\cdot; \lambda_k) = u(\cdot, \lambda_k)^T(1, \lambda_k)$, where $u(\cdot; \lambda_k)$ satisfies

$$\lambda_k^2 u(\cdot; \lambda_k) + \lambda_k B B^* u(\cdot; \lambda_k) + A u(\cdot; \lambda_k) = 0 \quad \text{with } u(\cdot, \lambda_k) \in H_1. \quad (2.3)$$

Since \mathcal{A}_B is real it follows that $\overline{W(\cdot; \lambda_k)} = W(\cdot; \overline{\lambda_k})$ is an eigenvector of \mathcal{A}_B corresponding to the eigenvalue $\overline{\lambda_k}$. We take the scalar product of the equation (2.3) with $u(\cdot, \lambda_k)$, we obtain:

$$\lambda_{\pm k} = -\frac{1}{2} \left\| B^* \left(\frac{u(\cdot; \lambda_k)}{\|u(\cdot; \lambda_k)\|_H} \right) \right\|_U^2 \pm \left(\frac{1}{4} \left\| B^* \left(\frac{u(\cdot; \lambda_k)}{\|u(\cdot; \lambda_k)\|_H} \right) \right\|_U^4 - \left\| A^{\frac{1}{2}} \left(\frac{u(\cdot; \lambda_k)}{\|u(\cdot; \lambda_k)\|_H} \right) \right\|_H^2 \right)^{\frac{1}{2}}.$$

Hence, if λ_k is a non-real eigenvalue, we find

$$\lambda_{\pm k} = -\frac{1}{2} \left\| B^* \left(\frac{u(\cdot; \lambda_k)}{\|u(\cdot; \lambda_k)\|_H} \right) \right\|_U^2 \pm i \sqrt{\left\| A^{\frac{1}{2}} \left(\frac{u(\cdot; \lambda_k)}{\|u(\cdot; \lambda_k)\|_H} \right) \right\|_H^2 - \frac{1}{4} \left\| B^* \left(\frac{u(\cdot; \lambda_k)}{\|u(\cdot; \lambda_k)\|_H} \right) \right\|_U^4},$$

which implies that, since B^* is bounded from H to U ,

$$0 < -\beta \leq \operatorname{Re}(\lambda_{\pm k}) = -\frac{1}{2} \left\| B^* \left(\frac{u(\cdot; \lambda_k)}{\|u(\cdot; \lambda_k)\|_H} \right) \right\|_U^2 \leq 0,$$

where $\beta := \frac{1}{2} \|B^*\|_{\mathcal{L}(H,U)}^2 < +\infty$, and

$$|\lambda_{\pm k}|^2 = \left\| A^{\frac{1}{2}} \left(\frac{u(\cdot; \lambda_k)}{\|u(\cdot; \lambda_k)\|_H} \right) \right\|_H^2 \geq \mu_1.$$

If λ_k is real we observe that

$$\sqrt{\frac{1}{4} \left\| B^* \left(\frac{u(\cdot; \lambda_k)}{\|u(\cdot; \lambda_k)\|_H} \right) \right\|_U^4 - \left\| A^{\frac{1}{2}} \left(\frac{u(\cdot; \lambda_k)}{\|u(\cdot; \lambda_k)\|_H} \right) \right\|_H^2} \leq (\beta^2 - \mu_1)_+^{\frac{1}{2}}.$$

Here $\mu_1 > 0$, is the first eigenvalue of A .

□

To state the principal result of this subsection (see Theorem 2.2), we need to introduce some notations. For $n \in \mathbb{N}^*$, we define the three complex numbers:

$$a_n = \sqrt{\mu_{n-1}} + \frac{1}{2}\delta_{n-1}, \quad b_n = \frac{1}{2}\delta_n + i\sqrt{\mu_n} \quad \text{and} \quad d_n = -\frac{1}{2}\delta_n + i\sqrt{\mu_n}, \quad (2.4)$$

where $\delta_k := \sqrt{\mu_{k+1}} - \sqrt{\mu_k}$ for $k \in \mathbb{N}^*$. Let $\text{Int}(\Gamma^{(n)})$ denote the rectangle with sides $\gamma_1^{(n)}, \gamma_2^{(n)}, \gamma_3^{(n)}$ and $\gamma_4^{(n)}$, (see Figure 1), where

$$\gamma_1^{(n)} := \{\lambda \in \mathbb{C}; \text{Im}(\lambda) = a_n \quad \text{and} \quad |\text{Re}(\lambda)| < \frac{\delta_n}{2}\},$$

$$\gamma_2^{(n)} := \{\lambda \in \mathbb{C}; \text{Re}(\lambda) = \frac{\delta_n}{2} \quad \text{and} \quad a_n \leq \text{Im}(\lambda) \leq a_{n+1}\},$$

$$\gamma_3^{(n)} := \{\lambda \in \mathbb{C}; \text{Im}(\lambda) = a_{n+1} \quad \text{and} \quad \text{Re}(\lambda) \text{ goes from } \frac{\delta_n}{2} \text{ to } -\frac{\delta_n}{2}\},$$

and

$$\gamma_4^{(n)} := \{\lambda \in \mathbb{C}; \text{Re}(\lambda) = -\frac{\delta_n}{2} \quad \text{and} \quad \text{Im}(\lambda) \text{ goes from } a_{n+1} \text{ to } a_n\}.$$

For $n = 1, 2, \dots$, we set

$$\Gamma^{(n)} = \gamma_1^{(n)} \cup \gamma_2^{(n)} \cup \gamma_3^{(n)} \cup \gamma_4^{(n)}, \quad \Gamma^{(-n)} := \{z \in \mathbb{C}; \bar{z} \in \Gamma^{(n)}\} \quad (2.5)$$

and

$$C^{(n)} = \{z \in \mathbb{C}; |\text{Im}(z)| < \sqrt{\mu_{n-1}} + \frac{\delta_{n-1}}{2} \text{ and } |\text{Re}(z)| < \frac{\delta_{n-1}}{2}\}.$$

Note that by construction $\text{Int}(\Gamma^{(k)}) \cap \text{Int}(\Gamma^{(n)}) = \emptyset$ for all $k, n \in \mathbb{Z}^*$ such that $k \neq n$. Here we denote the interior of $\Gamma^{(k)}$ by $\text{Int}(\Gamma^{(k)})$. Moreover, for all $N \in \mathbb{N}^*$ we have $\mathcal{C} \cup \mathcal{I} \subset C^{(N)} \bigcup_{|k| \geq N} \text{Int}(\Gamma^{(k)})$, where \mathcal{C} and \mathcal{I} are given by (2.1) and (2.2).

Theorem 2.2. *We assume (A1) and that (1.10) is satisfied. Then, there exists $N_0 \in \mathbb{N}^*$ large enough such that the operator \mathcal{A}_B has exactly $2N_0$ eigenvalues, including multiplicity, in C_{N_0} and one simple eigenvalue in $\text{Int}(\Gamma^{(k)})$ for each k with $|k| > N_0$. This exhausts the spectrum of \mathcal{A}_B .*

We have divided the proof into a sequence of lemmas.

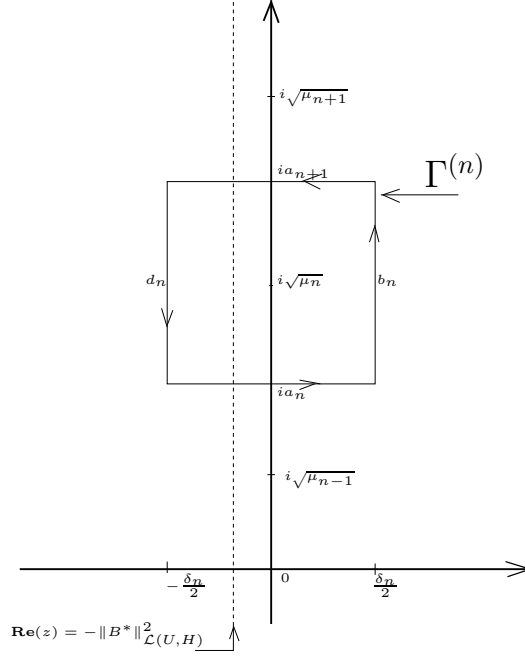
Lemma 2.3. *Assume (A1). Then, there exists $C > 0$ and $N_0 \in \mathbb{N}$ (large enough) such that for $n > N_0$, the following properties hold:*

$$(i) \quad \Gamma^{(\pm n)} \cup \partial C^{(n)} \subset \mathbb{C} \setminus (\sigma(\mathcal{A}_B) \cup \sigma(\mathcal{A}_0)).$$

(ii)

$$\|(\lambda - \mathcal{A}_B)^{-1} - (\lambda - \mathcal{A}_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{\delta_{n-1}^2}, \quad \text{uniformly on } \lambda \in \Gamma^{(\pm n)} \cup \partial C^{(n)}, \quad (2.6)$$

where $\partial C^{(n)}$ is the boundary of the rectangle $C^{(n)}$.

FIGURE 1. Location of $\sigma(\mathcal{A}_B)$

Proof. Since \mathcal{A}_0 is skew-adjoint, it follows that

$$\|(\lambda - \mathcal{A}_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{1}{\text{dist}(\lambda, \sigma(\mathcal{A}_0))}. \quad (2.7)$$

By construction of $\Gamma^{(\pm n)}$ and $C^{(n)}$, we have:

$$\begin{aligned} \text{dist}(\Gamma^{(\pm n)}, \sigma(\mathcal{A}_0)) &= \min(|b_n - i\sqrt{\mu_n}|, |d_n - i\sqrt{\mu_n}|, |ia_{n+1} - i\sqrt{\mu_{n+1}}|, |ia_n - i\sqrt{\mu_{n-1}}|) \\ &= \frac{\delta_{n-1}}{2}, \end{aligned}$$

and $\text{dist}(\partial C^{(n)}, \sigma(\mathcal{A}_0)) \geq \frac{\delta_{n-1}}{2}$, which together with (2.7) yields

$$\|(\lambda - \mathcal{A}_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{2}{\delta_{n-1}}, \quad \text{uniformly on } \lambda \in \Gamma^{(\pm n)} \cup \partial C^{(n)}. \quad (2.8)$$

Recalling that $\mathcal{B} = \mathcal{A}_0 - \mathcal{A}_B$ is a bounded linear operator on \mathcal{H} defined by

$$\mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & BB^* \end{pmatrix}.$$

From (2.8), we have

$$\|\mathcal{B}(\lambda - \mathcal{A}_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{2\|B^*\|_{\mathcal{L}(H,U)}^2}{\delta_{n-1}} \quad \text{uniformly on } \lambda \in \Gamma^{(\pm n)} \cup \partial C^{(n)}. \quad (2.9)$$

By **(A1)**, we choose N_0 such that for $n \geq N_0$:

$$\frac{2\|B^*\|_{\mathcal{L}(H,U)}^2}{\delta_{n-1}} \leq \kappa < 1.$$

Now the first statement of the lemma follows from (2.8), (2.9) and the following obvious equality:

$$\lambda - \mathcal{A}_B = [\text{Id} + \mathcal{B}(\lambda - \mathcal{A}_0)^{-1}](\lambda - \mathcal{A}_0). \quad (2.10)$$

On the other hand (2.10) yields

$$(\lambda - \mathcal{A}_B)^{-1} = (\lambda - \mathcal{A}_0)^{-1} + (\lambda - \mathcal{A}_0)^{-1} \sum_{p \geq 1} [-\mathcal{B}(\lambda - \mathcal{A}_0)^{-1}]^p,$$

which together with (2.8) and (2.9) imply (2.6). \square

According to Lemma 2.3, for $n \geq N_0$ the following Riesz projections are well defined:

$$\begin{aligned} P_{\Gamma(\pm n)}^B &:= \frac{1}{2\pi i} \int_{\Gamma(\pm n)} (\lambda - \mathcal{A}_B)^{-1} d\lambda, & P_{\Gamma(\pm n)}^0 &:= \frac{1}{2\pi i} \int_{\Gamma(\pm n)} (\lambda - \mathcal{A}_0)^{-1} d\lambda, \\ P_{\partial C^{(n)}}^B &:= \frac{1}{2\pi i} \int_{\partial C^{(n)}} (\lambda - \mathcal{A}_B)^{-1} d\lambda & \text{and} & P_{\partial C^{(n)}}^0 &:= \frac{1}{2\pi i} \int_{\partial C^{(n)}} (\lambda - \mathcal{A}_0)^{-1} d\lambda. \end{aligned} \quad (2.11)$$

The following result is a simple consequence of **(A1)**, (1.10), (2.6) and the estimate of the measure on $\partial C^{(n)}$, $\Gamma(\pm n)$.

Lemma 2.4. *We assume **(A1)** and that (1.10) is satisfied. Then, there exists $C > 0$ (independent of n) and $N_0 \in \mathbb{N}$ such that for $n \geq N_0$, we have*

$$\|P_{\Gamma(\pm n)}^B - P_{\Gamma(\pm n)}^0\|_{\mathcal{L}(\mathcal{H})} \leq C \frac{\delta_n}{\delta_{n-1}^2} < 1, \quad (2.12)$$

$$\|P_{\partial C^{(n)}}^B - P_{\partial C^{(n)}}^0\|_{\mathcal{L}(\mathcal{H})} \leq C \frac{\delta_n}{\delta_{n-1}^2} < 1. \quad (2.13)$$

End of the proof of Theorem 2.2: First, recalling that if P and Q are two projectors with $\|P - Q\| < 1$, then $\text{rank}(P) = \text{rank}(Q)$ (see Lemma 3.1 in [10]). Thus, in the notation of Lemma 3.3, we have

$$\text{rank}(P_{\partial C^{(n)}}^B) = \text{rank}(P_{\partial C^{(n)}}^0), \quad \text{rank}(P_{\Gamma(\pm n)}^B) = \text{rank}(P_{\Gamma(\pm n)}^0), \quad \text{for } n \geq N_0.$$

Next, we conclude from (2.1) and (2.2) that $\mathcal{C} \cup \mathcal{I} \subset C^{(N_0)} \cup \left(\bigcup_{|k| \geq N_0} \text{Int}(\Gamma^{(k)}) \right)$, hence that

$\sigma(\mathcal{A}_B)$ is a subset of $C^{(N_0)} \cup \left(\bigcup_{|k| \geq N_0} \text{Int}(\Gamma^{(k)}) \right)$. Now Theorem 2.2 follows from the fact that

$$\text{rank}(P_{\partial C^{(N_0)}}^0) = 2N_0 \quad \text{and} \quad \text{rank}(P_{\Gamma(\pm n)}^0) = 1.$$

\square

Remark 2.5. *In the proofs of Lemmas 2.3-2.4, we have used only the fact that the distance between two consecutive eigenvalues of \mathcal{A}_0 tends to infinity at infinity and the fact that \mathcal{A}_0 is a skew-adjoint operator. Similar general result are well-known (see Theorem 4.15a in [12]).*

2.2. Riesz basis. We start this subsection by constructing the eigenvectors associated to the high frequencies of \mathcal{A}_B . Since the high frequencies of \mathcal{A}_B are simple then for all $k \in \mathbb{N}^*$, $k > N_0$ (N_0 given by Theorem 2.2), we define

$$\varphi_{\pm k} = P_{\pm k}^B V_{\pm k}, \quad (2.14)$$

where $V_{\pm k}$ is the eigenvector of \mathcal{A}_0 associated to the eigenvalue $\pm i\sqrt{\mu_k}$ given by (1.6), and $P_{\Gamma(\pm k)}^B$ is given by (2.11). Note that $P_n^0 V_n = V_n$ for all $n \in \mathbb{Z}^*$.

For $n \in \mathbb{Z}^*$, we denote the eigenvalue of \mathcal{A}_B by $\lambda_n(B)$. We have the following proposition:

Proposition 2.6. *We assume (A1) and that (1.10) is satisfied. For $k \in \mathbb{Z}^*$, such that $|k| > N_0$, the function φ_k is an eigenvector of \mathcal{A}_B associated to the eigenvalue $\lambda_k := \lambda_k(\mathcal{A}_B)$. Moreover, there exists $C > 0$ such that*

$$\|\varphi_n - V_n\|_{\mathcal{H}} \leq C \frac{\delta_n}{\delta_{n-1}^2}, \quad \text{for all } n \text{ such that } |n| > N_0. \quad (2.15)$$

Here N_0 is given by Theorem 2.2.

In particular, $\|\varphi_n\|_{\mathcal{H}} = 1 + o(1)$ uniformly for $n \in \mathbb{Z}^*$, $|n| > N_0$.

Proof. For all $m \in \mathbb{Z}^*$, $|m| > N_0$, we have $\mathcal{A}_B \varphi_m = \mathcal{A}_B P_{\Gamma(m)}^B V_m = \lambda_m(B) P_{\Gamma(m)}^B V_m = \lambda_m(B) \varphi_m$. Using Lemma 2.4 and the fact that $P_{\Gamma(n)}^0 V_n = V_n$ with $\|V_n\|_{V \times L^2} = 1$, we get:

$$\|\varphi_m - V_m\|_{\mathcal{H}} = \|(P_{\Gamma(m)}^B - P_{\Gamma(m)}^0) V_m\|_{\mathcal{H}} \leq \|P_{\Gamma(m)}^B - P_{\Gamma(m)}^0\|_{\mathcal{L}(\mathcal{H})} \leq C \frac{\delta_m}{\delta_{m-1}^2},$$

for all $m \in \mathbb{Z}^*$, $|m| > N_0$, (C independent of m). In particular, parallelogram inequality and recalling that $\|V_m\|_{\mathcal{H}} = 1$ give that $\|\varphi_m\|_{\mathcal{H}} = 1 + o(1)$ uniformly for $m \in \mathbb{Z}^*$, $|m| > N_0$. \square

Now, we complete the sequence $(\varphi_k)_{|k| > N_0}$ of the eigenvectors associated to the high frequencies of \mathcal{A}_B by considering the generalized eigenvectors associated to the low frequencies of \mathcal{A}_B . Note that the number of these generalized eigenvectors associated to the low frequencies of \mathcal{A}_B is finite, at most $2N_0$ by Theorem 2.2. For $k \in \mathbb{Z}^*$ such that $|k| \leq N_0$, we denote by m_k the algebraic multiplicity of $\lambda_k := \lambda_k(\mathcal{A}_B)$ and we associated to it the Jordan chain of generalized eigenvectors, $(W_{k,p})_{p=0}^{m_k-1}$, i.e., a Jordan basis of the root subspace $\mathcal{E}_k := \{W \in \mathcal{H}; (\mathcal{A}_B - \lambda_k)^{m_k} W = 0\}$,

$$\mathcal{A}_B W_{k,l} = \lambda_k W_{k,l}, \quad \langle W_{k,l}, W_{k,l'} \rangle = 0, \quad l' < l = 0, \dots, p_k. \quad (2.16)$$

$$\begin{aligned} \mathcal{A}_B W_{k,m} &= \lambda_k W_{k,m} + W_{k,m-1}, \quad \langle W_{k,m}, W_{k,m'} \rangle = 0, \\ 0 \leq m' < m &= p_k + 1, \dots, m_k - 1. \end{aligned} \quad (2.17)$$

Here p_k is the dimension of the eigenspace $E_k := \{W \in \mathcal{H}; (\mathcal{A}_B - \lambda_k)W = 0\}$, $E_k \subset \mathcal{E}_k$.

Now, we take the family of generalized eigenvectors of \mathcal{A}_B :

$$\mathbb{B} := (W_{k,p})_{|k| \leq N_0, 0 \leq p \leq m_k - 1} \cup (\varphi_n)_{|n| > N_0}.$$

Since $\overline{\text{Vect}(\mathbb{B})} = \mathcal{H}$ (see Proposition 2.1, (iii)) and by assumption 1.10 the family \mathbb{B} is quadratically close to the orthonormal basis $(V_k)_{k \in \mathbb{Z}^*}$ of eigenvectors of the operator \mathcal{A}_0 (see (2.15)). Then it follows from the Fredholm Alternative, see e.g., [14, Appendix D, Theorem 3], the following result:

Theorem 2.7. *Assume (A1) and (A2). Then the set \mathbb{B} is a Riesz basis for the energy space \mathcal{H} . Moreover, there exists a linear isomorphism Φ of \mathcal{H} such that for all $n \in \mathbb{Z}^*$, $|n| > N_0$, $\Phi V_n = \varphi_n$ and $\Phi(\text{Vect}(V_n, |n| \leq N_0)) = \text{Vect}(W_{k,p}, |k| \leq N_0, 0 \leq p \leq m_k - 1)$.*

2.3. End of the proof of Main result. Using Theorem 2.7, we may expand the initial data as

$$[u^0, v^0] = \sum_{|k| \leq N_0} \sum_{p=0}^{m_k-1} c_{k,p} W_{k,p} + \sum_{|n| > N_0} c_n \varphi_n.$$

Then the solution of (1.2) is given by

$$[u, \partial_t u] = \sum_{|k| \leq N_0} \exp(\lambda_k t) \sum_{p=0}^{m_k-1} c_{k,p} \sum_{l=0}^p \frac{t^{p-l}}{(p-l)!} W_{k,l} + \sum_{|n| > N_0} c_n \exp(\lambda_n t) \varphi_n. \quad (2.18)$$

Recalling from Theorem 2.2 that at most $2N_0$ eigenvalues may be of algebraic multiplicity greater than one and that $2N_0$ is the maximum of such multiplicity, and the family $(V_{\pm k})_{k \in \mathbb{N}^*}$ is an orthonormal basis of the energy space \mathcal{H} (see Lemma 1.3), then, by the linear isomorphism Φ , we get

$$E(u(t)) = \left\| [u, \partial_t u] \right\|_{\mathcal{H}}^2 \leq \|\Phi\|^2 \|\Phi^{-1}\|^2 (1 + t^{2N_0}) \exp(2\mu(B)t) E(u(0)).$$

Then $\omega(B) \leq \mu(\mathcal{A}_B)$, this with inequality (1.9) we have established our main result. \square

Remark 2.8. *Note that, we talk about under (resp. over) damping if $\frac{1}{2}\|B^*\|_{\mathcal{L}(H,U)}^2$ is less (resp. greater) than $\sqrt{\mu_1}$, see [7]. Recall that $\mu_1 > 0$, is the first eigenvalue of A .*

3. SOME APPLICATIONS

Firstly, we give examples of dissipative systems which satisfy Assumption 1.10 and we deduce the main result for these samples. In the second part, we extend our result to some non-dissipative systems and we give an example that illustrates this situation.

3.1. Damped Euler-Bernoulli beam equation. We consider the following system:

$$\partial_t^2 u(x, t) + \partial_x^4 u(x, t) + 2a(x) \partial_t u(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \quad (3.1)$$

$$u(0, t) = u(1, t) = 0, \quad \partial_x^2 u(0, t) = \partial_x^2 u(1, t) = 0, \quad t > 0, \quad (3.2)$$

$$u(x, 0) = u^0(x), \quad \partial_t u(x, 0) = u^1(x), \quad 0 < x < 1, \quad (3.3)$$

where $a \in L^\infty(0, 1)$ is non-negative satisfying the following condition:

$$\exists c > 0 \text{ s.t., } a(x) \geq c, \text{ a.e., in a non-empty open subset } I \text{ of } (0, 1). \quad (3.4)$$

We define the energy of a solution u of (3.1)-(3.3), at time t , as

$$E(u(t)) = \frac{1}{2} \int_0^1 \left(|\partial_t u(x, t)|^2 + |\partial_x^2 u(x, t)|^2 \right) dx. \quad (3.5)$$

$$U = L^2(0, 1), \quad H = L^2(0, 1), \quad H_{\frac{1}{2}} = H^2(0, 1) \cap H_0^1(0, 1),$$

$$\mathcal{D}(A) = \left\{ u \in H^4(0, 1) \cap H_0^1(0, 1); \frac{d^2 u}{dx^2}(0) = \frac{d^2 u}{dx^2}(1) = 0 \right\},$$

$$\mathcal{H} = [H^2(0, 1) \cap H_0^1(0, 1)] \times L^2(0, 1),$$

$$A = \frac{d^4}{dx^4}, \quad B\phi = B^*\phi = \sqrt{2a(x)}\phi, \quad \forall \phi \in L^2(0, 1).$$

So,

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ -\frac{d^4}{dx^4} & 0 \end{pmatrix}, \quad \mathcal{A}_B = \begin{pmatrix} 0 & I \\ -\frac{d^4}{dx^4} & -2a(x) \end{pmatrix}.$$

- The operator \mathcal{A}_0 is skew-adjoint and with compact inverse and the spectrum is given by $\sigma(\mathcal{A}_0) = \{\pm i k^2 \pi^2, k \in \mathbb{N}^*\}$, then Assumptions **(A1)** and **(A2)** are satisfied.
- Note that the inequality (1.5) is satisfied according to [11], if a satisfies (3.4). So, $\omega(B) < 0$.
- As a direct implication of Theorem 1.4, we have the following result (this result was proved in [4]):

Proposition 3.1. *The fastest decay rate is given by the spectral abscissa, i.e.,*

$$\omega(B) = \mu(\mathcal{A}_B).$$

3.2. Extension to non-dissipative systems. We consider the system described by:

$$\ddot{x}(t) + Ax(t) + Kx(t) = 0, \quad (x(0), \dot{x}(0)) = (x_0, x_1) \in H_{\frac{1}{2}} \times H, \quad (3.6)$$

where A is the same operator as above and $K \in \mathcal{L}(H_{\frac{1}{2}}, H)$.

We can rewrite the system (3.6) as a first order differential equation, by putting $Y(t) = {}^T(x(t), \dot{x}(t))$:

$$\dot{Y}(t) + \mathcal{A}_K Y(t) = 0, \quad Y(0) = {}^T(x_0, x_1) \in \mathcal{H}, \quad (3.7)$$

where $\mathcal{A}_K := \mathcal{A}_0 - \mathcal{K} : \mathcal{D}(\mathcal{A}_K) = \mathcal{D}(\mathcal{A}_0) \subset \mathcal{H} \rightarrow \mathcal{H}$, with

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} : \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(A) \times H_{\frac{1}{2}} \subset \mathcal{H} \rightarrow \mathcal{H},$$

and $\mathcal{K} = \begin{pmatrix} 0 & 0 \\ -K & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H})$.

The system (3.6) is well-posed. More precisely, the following classical result holds.

Proposition 3.2. *Suppose that $(x_0, x_1) \in \mathcal{H}$. Then the problem (1.1) admits a unique solution x in the following space $C([0, +\infty); H_{\frac{1}{2}}) \cap C^1([0, +\infty); H)$.*

We denote,

$$E(x(t)) = \frac{1}{2} \left\| (x(t), \dot{x}(t)) \right\|_{\mathcal{H}}^2.$$

Let $\mu(K)$ be the *spectral abscissa* of \mathcal{A}_K given by:

$$\mu(K) = \sup \{ \operatorname{Re}(\lambda); \lambda \in \sigma(\mathcal{A}_K) \}. \quad (3.8)$$

Here $\sigma(\mathcal{A}_K)$ denotes the spectrum of \mathcal{A}_K .

We define the growth bound, depending on K , as

$$\omega(K) = \inf \{ \omega; \text{there exists } C = C(\omega) > 0 \text{ such that} \}$$

$$E(x(t)) \leq C(\omega) e^{2\omega t} E(x(0)) \text{ for every solution of (3.6) with initial data in } \mathcal{H} \}. \quad (3.9)$$

As above we can prove the following result.

Theorem 3.3. *Assume (A1) and (A2). Then,*

- (i) *The eigenvectors of the associated operator \mathcal{A}_K form a Riesz basis in the energy space \mathcal{H}*
- (ii)

$$\omega(K) = \mu(\mathcal{A}_K). \quad (3.10)$$

Example : *Euler-Bernoulli equation with force term*

We consider the following initial and boundary value problem:

$$\partial_t^2 u(x, t) + \partial_x^4 u(x, t) + p \partial_x^2 u(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \quad (3.11)$$

$$u(0, t) = \partial_x u(0, t) = 0, \quad \partial_x^2 u(1, t) = 0, \quad \partial_x^3 u(1, t) = 0, \quad t > 0, \quad (3.12)$$

$$u(x, 0) = u^0(x, 0), \quad \partial_t u(0, x) = u^1(x), \quad 0 < x < 1, \quad (3.13)$$

where p is a positive constant.

Here,

$$H = L^2(0, 1), \quad H_{\frac{1}{2}} = \left\{ u \in H^2(0, 1); u(0) = 0, \frac{du}{dx}(0) = 0 \right\},$$

and the operators are defined

$$\mathcal{A}_0 = \begin{pmatrix} 0 & Id \\ -\frac{d^4}{dx^4} - p \frac{d^2}{dx^2} & 0 \end{pmatrix},$$

$$\mathcal{D}(\mathcal{A}_0) = \left\{ (u, v) \in \left(H^4(0, 1) \cap H_{\frac{1}{2}} \right) \times H_{\frac{1}{2}}, \frac{d^2 u}{dx^2}(1) = 0, \frac{d^3 u}{dx^3}(1) = 0 \right\},$$

and $K = p \frac{d^2}{dx^2} \in \mathcal{L}(H_{\frac{1}{2}}, H)$.

We have the for all $(u^0, u^1) \in H_{\frac{1}{2}} \times L^2(0, 1)$ the problem (3.11)-(3.13) admits a unique solution

$$u \in C([0, +\infty); H_{\frac{1}{2}}) \cap C^1([0, +\infty); L^2(0, 1)).$$

The spectrum of \mathcal{A}_0 is given by $(\pm i k^2 \pi^2)_{k \in \mathbb{N}^*}$. Then Assumptions **(A1)** and **(A2)** are satisfied. Therefore, according to Theorem 3.3, we obtain

Proposition 3.4.

- (i) *The generalized eigenvectors of the associated operator \mathcal{A}_K form a Riesz basis in the energy space \mathcal{H} .*
- (ii) $\omega(K) = \mu(\mathcal{A}_K)$.

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